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Expansion of Gas Clouds
and Hypersonic Jets
Bounded by a Vacuum

SEPTEMBER 1962

Prepared by

H. MIRELS and J. F. MULLEN

Aerodynamics and Propulsion Research Laboratory

Prepared for COMMANDER SPACE SYSTEMS DIVISION

UNITED STATES AIR FORCE

Inglewood, California

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Report No.
TDR-169(3230-12)TR-1

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BOUNDED BY A VACUUM

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ABSTRACT

The unsteady expansion of gas clouds, bounded by a vacuum, is considered for both continuum and free molecular flows. Known analytical solutions are reviewed with emphasis on the asymptotic nature of the flow after long times. An approximate analytical solution is presented for the continuum expansion, after long time, of an initially uniform gas cloud. For a plane expansion, the approximate solution is in exact agreement with the known exact solution. For cylindrical and spherical expansions, the approximate solution gives results for density (at the axis) which are about 10 percent higher than those obtained by a numerical integration of the equations of motion. (Only two numerical solutions are available for the latter comparison.) The continuum and free molecular expansions of an initially uniform gas are found to be quite similar, after long time, except for the flow in the vicinity of the leading edge of the expansion. For a given gas, the density at the axis decays more rapidly, with time, for the continuum case. The extension of these results, to hypersonic jets, bounded by a vacuum, is also noted.

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NOMENCLATURE

a	-	sound speed
F	-	hypergeometric function
h	-	$\gamma/2a_0^2$
M	-	Mach number
m	-	elemental mass
N	-	$(\gamma+1)/2(\gamma-1)$
p	-	pressure
R	-	ordinate of leading edge of expansion
r	-	spatial coordinate, transverse for jet flows
t	-	time
u	-	axial velocity for jet flow
v	-	velocity, transverse for jet flows
x	-	axial coordinate for jet flows
α, β	-	characteristic Riemann invariants
Γ	-	gamma fraction
γ	-	specific heat ratio
δ	-	characteristic streamline slope
η	-	similarity variable
ρ	-	density
σ	-	dimensional index (0, 1, 2)

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INTRODUCTION

Space flight has intensified interest in the unsteady expansion of gas clouds into a vacuum. A related problem, the lateral expansion of a hypersonic jet bounded by a vacuum, is also of current interest.

Several analytic solutions exist for unsteady expansions into a vacuum. References 1 and 2 have treated plane, cylindrical and spherical self similar flows. These are flows where time can be eliminated from the equations of motion, reducing them to ordinary differential equations which can be integrated in closed form. Only special initial conditions, for a gas cloud, can lead to self similar solutions. In particular, the initial density distribution can not be uniform. The plane problem of an initially uniform gas cloud, expanding into a vacuum, has been solved analytically in Reference 3. The solution for an initially uniform cylindrical or spherical gas cloud requires numerical integration of the equations of motion. The above studies use continuum equations of motion. If the mean free path in a gas cloud is of the order (or larger) of the cloud diameter, the resulting expansion can be treated as a free molecule flow. Reference 4 has obtained solutions, in quadrature form, for the free molecular expansion of arbitrary gas clouds.

In the present study, the asymptotic nature of the flow field, after long times, is examined for each of the analytical solutions mentioned above. These are used as a guide to develop an approximate asymptotic solution for the expansion of an initially uniform cylindrical or spherical gas cloud. Reference 5 has discussed the asymptotic nature of the flow, near the origin, for the plane problem of an initially uniform gas cloud expanding into a vacuum. The present study is a generalization of Reference 5 in that the entire flow field between the origin and the leading edge of the expansion is considered. In addition, cylindrical and spherical expansions are considered. The application of the unsteady plane and cylindrical flow results for finding the lateral spreading of a steady hypersonic jet, bounded by a vacuum, is also discussed.

UNSTEADY EXPANSION OF GAS INTO VACUUM

General Considerations

The equations of motion for unsteady flows in one spatial variable, \bar{r} , may be written

$$\text{Continuity:} \quad \frac{\partial \bar{\rho}}{\partial \bar{t}} + \frac{\partial \bar{\rho} \bar{v}}{\partial \bar{r}} + \sigma \frac{\bar{\rho} \bar{v}}{\bar{r}} = 0 \quad (1a)$$

$$\begin{aligned} \text{Momentum:} \quad \frac{\partial \bar{v}}{\partial \bar{t}} + \bar{v} \frac{\partial \bar{v}}{\partial \bar{r}} = - \frac{1}{\bar{\rho}} \frac{\partial \bar{p}}{\partial \bar{r}} \\ - \gamma c \bar{\rho}^{-(\gamma-2)} \frac{\partial \bar{\rho}}{\partial \bar{r}} \end{aligned} \quad (1b)$$

when the superscript bar indicates a dimensional quantity and $\sigma = 0, 1, 2$ for plane, cylindrical and spherical flows, respectively. See Nomenclature for definition of the other variables. Eq. 1b and all subsequent developments, assume that the entropy of the gas is uniform (i. e. the relation $\bar{p} = c \bar{\rho}^\gamma$, where c is a constant, has been used to eliminate the variable \bar{p} in Eq. 1b so that there are now two dependent variables, $\bar{\rho}$ and \bar{v}).

Eq. 1 can be nondimensionalized by introducing the following variables

$$\rho = \frac{\bar{\rho}}{\bar{\rho}_0} \quad v = \frac{\bar{v}}{\bar{a}_0} \quad r = \frac{\bar{r}}{\bar{R}_0} \quad t = \frac{\bar{t} \bar{a}_0}{\bar{R}_0} \quad (2)$$

In all subsequent developments, \bar{R}_0 will be taken to be the initial ($\bar{t} = 0$) gas cloud radius. For initially uniform gases, $\bar{\rho}_0$ and \bar{a}_0 are the initial density

and sound speed, respectively. For initially non-uniform gases, $\bar{\rho}_0$ and \bar{a}_0 are evaluated at $\bar{r} = 0$, $\bar{t} = 0$. Substitution of Eqs. 2 into Eqs. 1 yields

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho v}{\partial r} + \sigma \frac{\rho v}{r} = 0 \quad (3a)$$

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial r} = - \rho^{-\gamma-2} \frac{\partial \rho}{\partial r} \quad (3b)$$

which are the non-dimensional equations considered herein.

As pointed out by Reference 6, the pressure gradient term in the momentum equation (right side of Eq. 3b) becomes negligible, after long times, when studying the expansion of gases into a vacuum. Eq. 3b can then be integrated to yield

$$v = r/t \quad (4)$$

Reference 6 refers to such flows as "inertia dominated". That is, each gas particle moves with a constant velocity and is unaffected by the other particles. Substitution of Eq. 4 into Eq. 3a shows that the density distribution must have the form

$$\rho t^{\sigma+1} = f(r/t) \quad (5)$$

where $f(r/t)$ is a function of r/t . The precise form for this function depends on the earlier motion of the gas cloud. The motion is self similar, after long times, since v and $\rho t^{\sigma+1}$ are functions only of r/t . All solutions discussed herein will have this asymptotic behavior.

Self Similar Flows

References 1 and 2 have investigated expansions into a vacuum which are self similar at all times. These flows will now be discussed.

Let $R(t)$ be the leading edge of a gas cloud which is expanding into a vacuum. The gas particles are then contained in the region $0 \leq r \leq R(t)$. A vacuum exists for $r > R(t)$. Define a similarity variable

$$\eta = r/R(t) \quad (6)$$

so that $\eta = 1$ corresponds to the leading edge of the gas cloud and the gas is confined to the region $0 \leq \eta \leq 1$. Self similar solutions are found by assuming the dependent variables have the form $v = g(t)\phi(\eta)$ and $\rho = h(t)f(\eta)$, substituting into Eqs. 3, and finding the form of $g(t)$ and $h(t)$ such that the resulting equations are independent of t . The resulting solution, for a gas expanding into a vacuum, is (Refs. 1 and 2)

$$\rho = R^{-(\sigma+1)} [1-\eta^2]^{\frac{1}{\gamma-1}} \quad (7a)$$

$$v = \left(\frac{dR}{dt}\right) \eta \quad (7b)$$

where

$$\frac{dR}{dt} = \frac{1}{\sqrt{\sigma+1}} \frac{2}{\gamma-1} \sqrt{1-R^{-(\sigma+1)(\gamma-1)}} \quad (8)$$

This solution satisfies the boundary conditions:

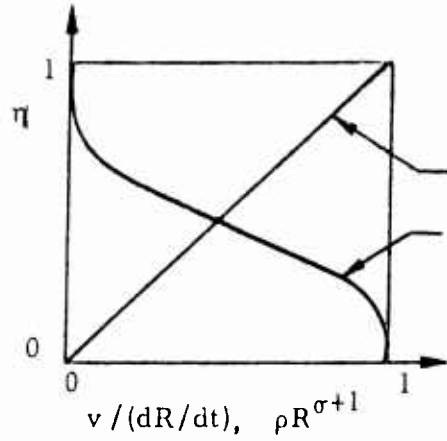
$$\text{at } t = 0; \quad R = 1, \quad \frac{dR}{dt} = 0 \quad (9a)$$

$$t = 0, \quad r = 0; \quad \rho = 1 \quad (9b)$$

$$t \geq 0, \quad \eta = 1; \quad \rho = 0 \quad (9c)$$

$$t \geq 0, \quad \eta = 0; \quad v = 0 \quad (9d)$$

Thus, initially ($t = 0$), $v = 0$, and ρ varies from 1 at $r = 0$ to 0 at $r = R$. At later times, v is not zero, and varies linearly with r .



The variation of $v/(dR/dt)$ and $\rho R^{\sigma+1}$ with η is illustrated in Sketch 1.

Sketch 1

Variation of $v(dR/dt)$ and $\rho R^{\sigma+1}$ with η

The location of the leading edge of the expansion, at any time, can be found by integrating Eq. 8, which can be put in the form

$$\sqrt{\frac{1}{\sigma+1} \frac{2}{\gamma-1}} t = \int_1^R \frac{dR}{\sqrt{1-R^{-(\sigma+1)(\gamma-1)}}} \quad (10a)$$

$$= N_0 \int_0^{\sec^{-1} R^{1/N_0}} (\sec \Theta)^{1+N_0} d\Theta \quad (10b)$$

where $N_0 \equiv 2/(\sigma+1)(\gamma-1)$. Eq. 10b is obtained from Eq. 10a by letting $R = \sec^{N_0} \theta$. Eq. 10b can be integrated in closed form when N_0 is an integer. For example

$$\frac{1}{\sqrt{\sigma+1}} \frac{2}{\gamma-1} t = \sqrt{R^2 - 1} \quad \text{for } N_0 = 1 \quad (11a)$$

$$= \sqrt{R(R-1)} + \ln(\sqrt{R-1} + \sqrt{R}) \quad \text{for } N_0 = 2 \quad (11b)$$

$$= (R^{2/3} + 2) \sqrt{R^{2/3} - 1} \quad \text{for } N_0 = 3 \quad (11c)$$

Eq. 11a includes the case $\sigma = 2, \gamma = 5/3$. Eq. 11b includes the cases $\sigma = 1, \gamma = 3/2$ and $\sigma = 2, \gamma = 4/3$. Eq. 11c includes the cases $\sigma = 0, \gamma = 5/3$; $\sigma = 1, \gamma = 4/3$ and $\sigma = 2, \gamma = 11/9$. Thus Eq. 10b permits the complete closed form solution for a large number of cases of interest. The reduction of Eq. 10a to Eq. 10b does not appear to have been presented previously. The variation of R with t is illustrated in Fig. 1 which presents results for $\gamma = 4/3, 5/3$ and $\sigma = 0, 1$ and 2 .

The asymptotic nature of the flow, after long times, will now be noted. Eq. 10a shows that for $R^{-(\sigma+1)(\gamma-1)} \ll 1$ (i. e. after long times).

$$R = \frac{1}{\sqrt{\sigma+1}} \frac{2}{\gamma-1} t \quad (12)$$

Thus, after long times, the leading edge moves with a uniform velocity $\frac{dR}{dt} = \frac{2}{\sqrt{\sigma+1}(\gamma-1)}$. For $\sigma = 0$, this velocity is the same as that associated with

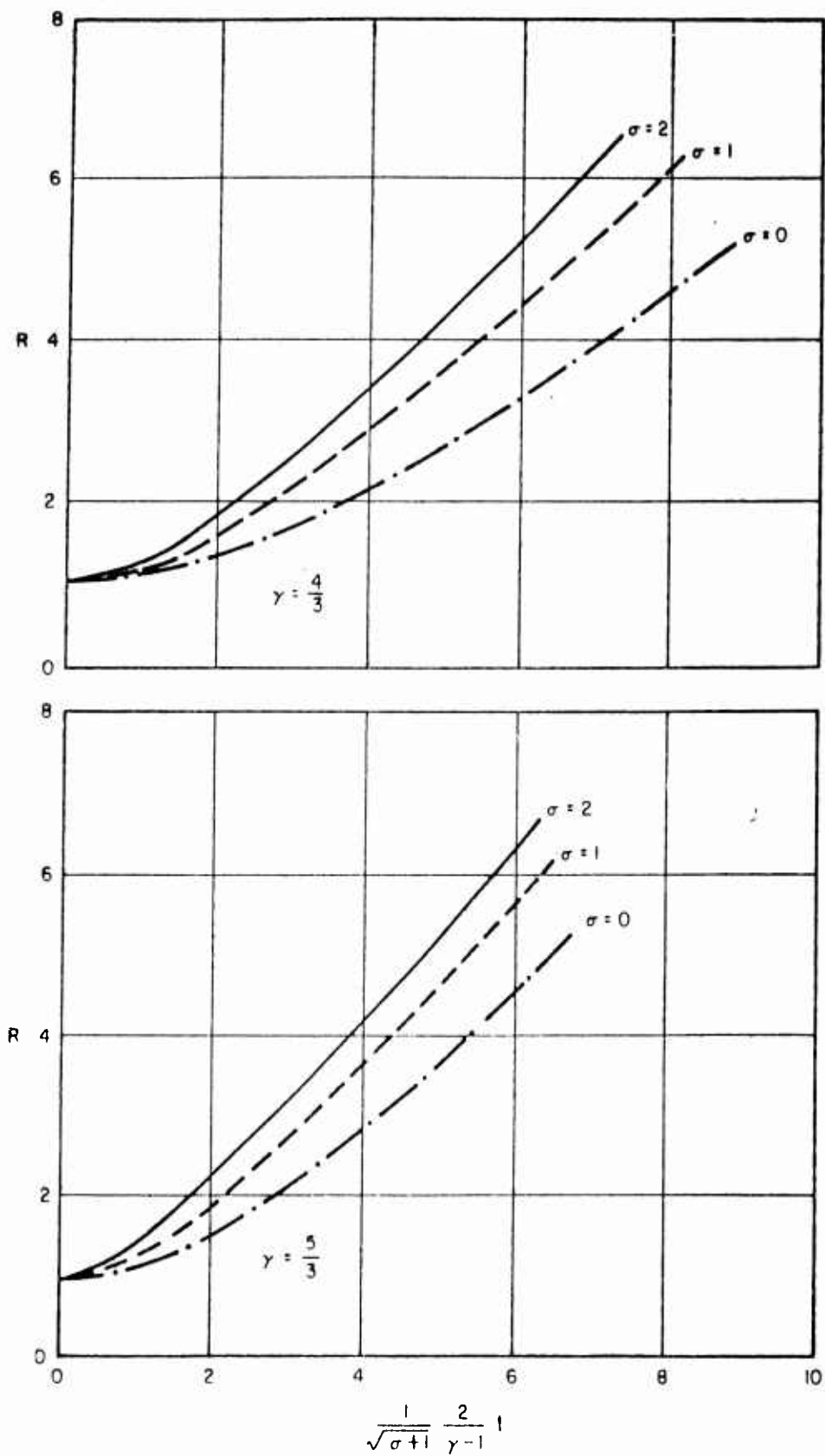


Figure 1. Leading Edge of Expansion for Self Similar Flows
(From Ref. 1, pg. 276)

the expansion of an initially uniform gas cloud into a vacuum⁽¹⁾ However, for $\sigma = 1, 2$, the final velocity is less, by the factor $1/\sqrt{\sigma+1}$, than the corresponding velocity for an initially uniform gas cloud. The reason for this is that in the present case the particles at the leading edge of the expansion have no energy at $t = 0$ and are subsequently accelerated by the particles in the interior of the gas cloud. For $\sigma = 0$, there are sufficient particles at and near the energy corresponding to conditions at $t = r = 0$, to drive the front to the limiting velocity $dR/dt = 2/(\gamma-1)$. With increase in σ , there is a decrease in the relative number of particles which have energy at or near the value at $t = r = 0$. Hence, the front is accelerated to lower velocity as σ increases.

After long times, the velocity and density are given by (from Eqs 7 and 12)

$$v = r/t \quad (13a)$$

$$\rho t^{\sigma+1} = \left(\frac{\sqrt{\sigma+1} (\gamma-1)}{2} \right)^{\sigma+1} (1-\eta^2)^{1/(\gamma-1)} \quad (13b)$$

where $\eta = \sqrt{\sigma+1} \frac{\gamma-1}{2} \frac{r}{t}$. These expressions are in agreement with Eqs 4 and 5.

Analytical Solution for $\sigma = 0$ and Uniform Initial Density

Ref. 3 has given the characteristics solution for the plane ($\sigma = 0$) expansion into a vacuum of a gas which is initially uniform. This solution has

⁽¹⁾ For an initially uniform gas cloud, $dR/dt = 2/(\gamma-1)$, for all $t > 0$, regardless of the value of σ . (The case of an initially uniform gas and $\sigma = 0$ is treated in the next section). This is true because at $t = 0$ the flow in the vicinity of $r = R$ behaves as through $\sigma = 0$. The leading edge achieves the limiting velocity $dR/dt = 2/(\gamma-1)$ and is unaffected by the subsequent expansion of the remainder of the gas. See Ref. 7 for further discussion.

been discussed in greater detail in Refs. 5 and 6. The uniform gas originally extends from $r = 0$ to $r = 1$ where $r = 0$ can correspond to a wall or to a plane of symmetry. The expansion is assumed to start at $t = 0$. At first a simple expansion wave is generated which is centered at $r = 1$, $t = 0$. This region is bounded by the lines $\beta = -1$, $\beta = 1$ and $\alpha = 1$ in Fig. 2. The reflection of this wave, at $r = 0$, results in an interaction region, also indicated in Fig. 2. The solution for both regions is discussed herein. In this discussion it is convenient to use the local speed of sound $a = \bar{a}/\bar{a}_0$, instead of ρ , as a dependent variable. These are related by, for isentropic flow,

$$\rho = a^{2/(\gamma-1)} \quad (14)$$

In the simple wave region, the dependent variables depend only on $(r - 1)/t$. The solution of Eqs. 3 for v and a can then be shown to be

$$v = \frac{2}{\gamma+1} \left[1 + \frac{r-1}{t} \right] \quad (15a)$$

$$a = \frac{2}{\gamma+1} \left[1 - \frac{\gamma-1}{2} \frac{r-1}{t} \right] \quad (15b)$$

The leading edge of the simple wave corresponds to $a = 0$ and its location is given by $R = r = 1 + 2t/(\gamma-1)$. The leading edge moves with velocity $dR/dt = 2/(\gamma-1)$ as previously noted.⁽¹⁾ The trailing edge of the simple wave is defined by the line $r = (1 - t)$. The fluid between this line and $r = 0$ is uniform at the initial values, $v = 0$, $\rho = a = 1$.

The solution for the interaction region requires the introduction of the Riemann variables

$$\alpha = a + \frac{\gamma-1}{2} v \quad (16a)$$

$$\beta = a - \frac{\gamma-1}{2} v \quad (16b)$$

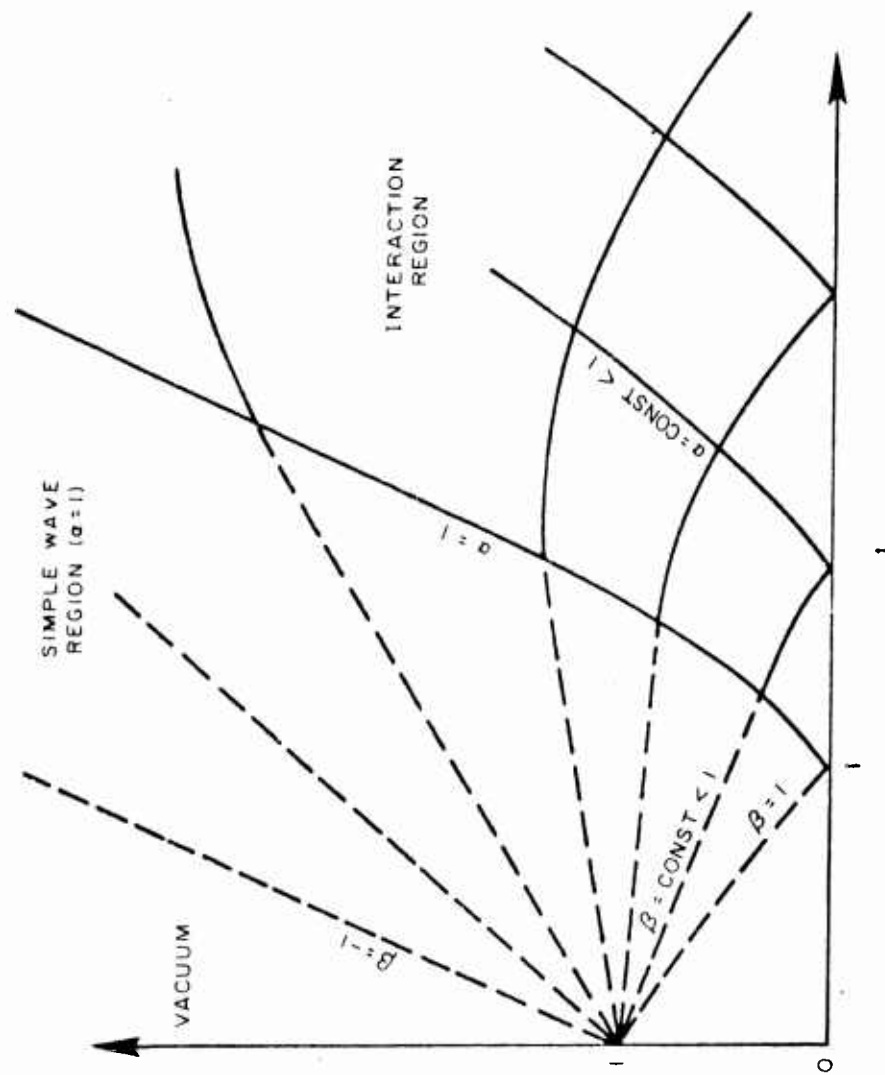


Figure 2. Lines of Constant α and β (Characteristics) for Expansion of Uniform Gas into a Vacuum, $\sigma = 0$.

The quantities α and β are constant along characteristic lines which propagate with velocity $v + a$ and $v - a$, respectively. These characteristic lines are illustrated in Fig. 2. (In the simple wave region $\alpha = 1$ while β varies from -1 at the leading edge to $+1$ at the trailing edge). From Eqs. 16 it follows that

$$v = \frac{1}{\gamma-1} (\alpha - \beta) \quad (17a)$$

$$a = \frac{1}{2} (\alpha + \beta) \quad (17b)$$

With these relations, Eqs. 3 can be inverted so that (r, t) are considered functions of (α, β) . The resulting Eqs. are

$$\frac{\partial r}{\partial \beta} = \frac{1}{\gamma-1} \left(\frac{\gamma+1}{2} \alpha + \frac{\gamma-3}{2} \beta \right) \frac{\partial t}{\partial \beta} \quad (18a)$$

$$\frac{\partial r}{\partial \alpha} = \frac{1}{\gamma-1} \left(\frac{3-\gamma}{2} \alpha - \frac{\gamma+1}{2} \beta \right) \frac{\partial t}{\partial \alpha} \quad (18b)$$

Eliminating r yields a single second order partial differential equation

$$\frac{\partial^2 t}{\partial \alpha \partial \beta} + \frac{N}{\alpha + \beta} \left(\frac{\partial t}{\partial \alpha} + \frac{\partial t}{\partial \beta} \right) = 0 \quad (19a)$$

where

$$N = \frac{\gamma+1}{2(\gamma-1)} \quad (19b)$$

The boundary between the simple wave region and the interaction region (line $\alpha = 1$ in Fig. 2) is defined by (see Refs 3, 5 or 6)

$$t(1, \beta) = \left(\frac{2}{1+\beta}\right)^N \quad (20)$$

where β varies from 1 to -1 as t goes from 1 to ∞ . An explicit expression for $t(\alpha, \beta)$ in the interaction region is found by integrating Eq. 19a, with the initial conditions given by Eq. 20, and equals

$$t(\alpha, \beta) = \left(\frac{2}{\alpha+\beta}\right)^N F(1-N, N, 1; -\frac{(1-\alpha)(1-\beta)}{2(\alpha+\beta)}) \quad (21)$$

Here, $F(\)$ is the hypergeometric function. If N is a positive integer, the hypergeometric function in Eq. 21 reduces to a polynomial of degree $N - 1$. In particular, for N a positive integer

$$F(1-N, N, 1; z) = 1 + \sum_{i=1}^{N-1} \frac{\Gamma(N+1)}{\Gamma(N-i) \Gamma(i+1)} \frac{(-z)^i}{2} \quad (22)$$

where $\Gamma(\)$ is the gamma function ($\Gamma(N+1) = N\Gamma(N)$, $\Gamma(1) = 1$, $\Gamma(1/2) = \sqrt{\pi}$). Eqs. 18a or 18b can then be integrated to give $r(\alpha, \beta)$. Thus, for N an integer, an analytic solution for the interaction region is obtained. Note, from Eq. 19b, that $\gamma = (2N+1)/2N-1$ so that $\gamma = 3, 5/3, 7/5, \dots$ for $N = 1, 2, 3, \dots$

The variation, with time, of the fluid properties at $r = 0$ can be found from Eq. 21. At $r = 0$, $\alpha = \beta = a$ so that Eq. 21 relates t and a (or ρ, p, T) for $t \geq 1$. The variation of ρ with t , at $r = 0$, is plotted in Fig. 3 for $\gamma = 5/3, 7/5, 9/7, 11/9$. The asymptotic solution, for long times, is indicated by a dashed line in these figures. It is seen that the flow approaches the asymptotic solution more slowly as γ decreases.

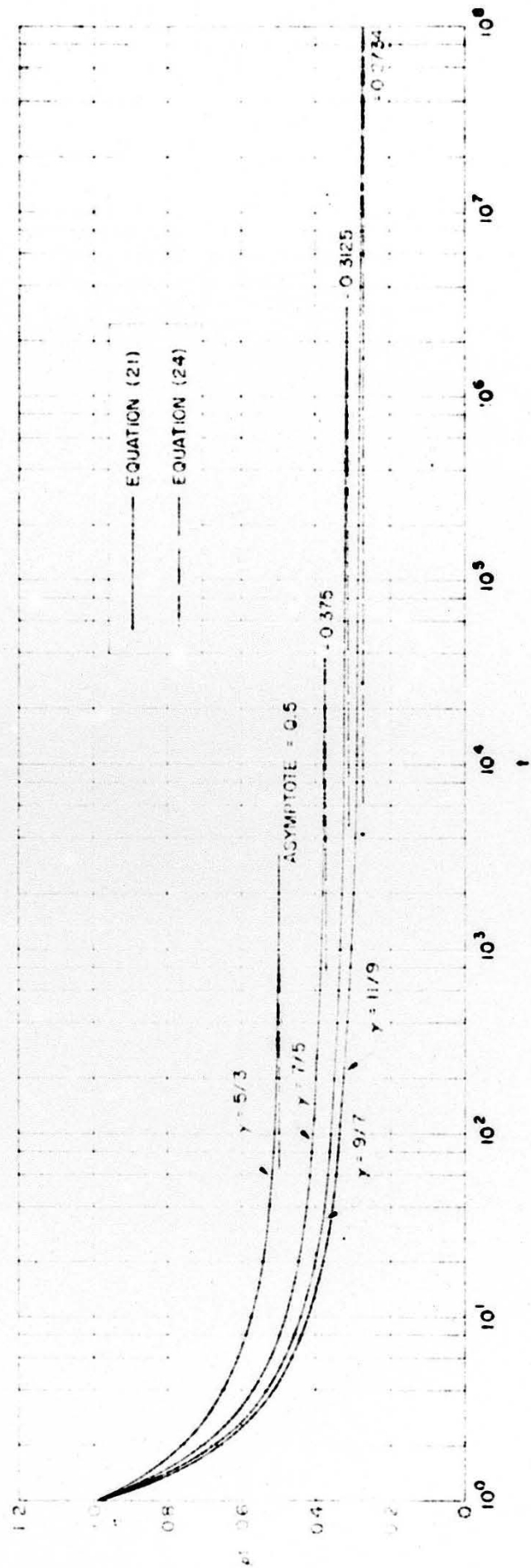


Figure 3. Variation of Density with Time, at $r = 0$, for Plane ($n = 0$) Expansion of Initially Uniform Gas into Vacuum.

The entire flow field, after long times, can also be described. The velocity field is given by $v = r/t$ (Eq. 4). The density distribution, and other state properties, are found from Eq. 21. Note that $\alpha + \beta = 2\alpha$ approaches zero as $t \rightarrow \infty$. Hence $(1 - \alpha)(1 - \beta)/2(\alpha + \beta) \rightarrow \infty$, as $t \rightarrow \infty$, provided $\alpha < 1$, and only the last term need be retained in the polynomial expansion for $F(\)$ in Eq. 22. This results in the following expression for ρ :

$$\rho t = \frac{(\gamma-1)}{\sqrt{\pi}} \frac{\Gamma(N+1/2)}{\Gamma(N)} \left[1 - \left(\frac{\gamma-1}{2} \frac{r}{t} \right)^2 \right]^{\frac{3-\gamma}{2(\gamma-1)}} \quad (23)$$

Eq. 23 is valid for N an integer, $t \rightarrow \infty$ and $\alpha < 1$.

After long times, the characteristic line $\alpha = 1$ propagates with velocity $2/(\gamma-1)$ (from Eq. 16a with $\alpha = 1$, $\alpha = 0$) and is thus parallel to the leading edge of the simple wave ($\beta = -1$ in Fig. 2). With increasing time, the mass contained between these lines becomes negligible, (i. e. the simple wave region becomes negligible) and the leading edge of the expansion can be approximated by $R = 2t/(\gamma-1)$. Again defining $\eta \equiv r/R$, Eq. 23 then becomes

$$\rho t = \frac{(\gamma-1)}{\sqrt{\pi}} \frac{\Gamma(N+1/2)}{\Gamma(N)} (1-\eta^2)^{\frac{3-\gamma}{2(\gamma-1)}} \quad (24)$$

which may be considered valid for $t \rightarrow \infty$ and $0 \leq \eta \leq 1$. Eq. 24 is quite similar in form to Eq. 13b with $\sigma = 0$.

The asymptotic solution noted in Fig. 3 is obtained from Eq. 24 with $\eta = 0$

Approximate Asymptotic Solution for Uniform Initial Density.

Eq. 24 gives the asymptotic nature of the flow, as $t \rightarrow \infty$, for a plane expansion ($\sigma = 0$) of an initially uniform gas cloud. It would be of interest to develop corresponding expressions for cylindrical and spherical expansions of an initially uniform gas cloud. Approximate expressions, which satisfy conservation of mass and energy, will now be developed.

Eqs. 13b and 24 suggest that a suitable form for ρ , as $t \rightarrow \infty$, is

$$\rho t^{\sigma+1} = D (1-\eta)^2 B \quad (25)$$

where B and D are constants and $\eta = r/R$. Since an initially uniform gas cloud is being considered, R will be taken as

$$R = \frac{2}{\gamma-1} t \quad (26)$$

The constants B and D can be found from conservation of mass and energy. The latter can be expressed as

$$\frac{1}{\sigma+1} = R^{\sigma+1} \int_0^1 \rho \eta^{\sigma} d\eta \quad (27a)$$

$$\frac{1}{\gamma(\gamma-1)(\sigma+1)} = R^{\sigma+1} \int_0^1 \rho \frac{v^2}{2} \eta^{\sigma} d\eta \quad (27b)$$

Eq. 27a equates the mass at $t = 0$ to the mass at $t \rightarrow \infty$ and Eq. 27b equates the internal energy at $t = 0$ to the kinetic energy at $t \rightarrow \infty$. (The internal energy $\rightarrow 0$ as $t \rightarrow \infty$). Substituting Eqs. 25 and 26 in Eqs. 27, with $v = r/t$, gives

$$B = \frac{\sigma+1}{2} \left(\frac{\gamma+1}{\gamma-1} \right) - 1 \quad (\sigma+1) N - 1 \quad (28a)$$

$$D = \frac{2}{\sigma+1} \left(\frac{\gamma-1}{2} \right)^{\sigma+1} \frac{\Gamma(B+1 + \frac{\sigma+1}{2})}{\Gamma(B+1) \Gamma(\frac{\sigma+1}{2})} \quad (28b)$$

For $\sigma = 1$ and N an integer, $D = (\gamma^2 - 1)/4$. Numerical values for B and D are given in Table I for $\sigma = 0, 1, 2$ and $\gamma = 5/3, 7/5, 9/7, 11/9$ and $13/11$.

Eqs. 25 and 28 are in exact agreement with Eq. 24 for $\sigma = 0$ and N an integer. Since Eqs. 25 and 28 are an approximate integral method solution, they may be considered valid for N not an integer, as well as for $\sigma = 0, 1, 2$. A similar solution has been presented by Ref. 6, among a number of other approximate solutions.

The variation of density, with η , is given in Fig. 4 for $\gamma = 7/5$ and $\sigma = 0, 1, 2$. It is seen that as σ increases, the density decreases more rapidly with η . The variation of density with η is given in Fig. 5 for $\sigma = 1$ and $\gamma = 5/3, 7/5$ and $9/7$. The density decreases more rapidly with η as γ decreases.

Free Molecular Expansion

If the mean free path⁽²⁾ of the molecules in a gas is of order \bar{R}_0 , or larger, the expansion may be treated as a free molecule flow. That is, collisions between molecules can be neglected and each molecule moves with a constant velocity throughout the expansion. This case has been discussed in Ref. 4 for various initial density distributions. The free molecular expansion of initially uniform gas clouds is discussed herein for $t \rightarrow \infty$ and the results are compared with those from continuum theory.

Consider an elemental mass \bar{m}_0 of particles to be located at \bar{r}_0 at $\bar{t} = 0$ and to have a Maxwellian distribution of velocities. The mass of particles having velocities between \bar{v} and $\bar{v} + \Delta\bar{v}$ is denoted by $\Delta\bar{m}_0$ and equals

$$\frac{\Delta\bar{m}_0}{\bar{m}_0} = 4\pi \left(\frac{h}{\pi}\right)^{3/2} e^{-h\bar{v}^2/\bar{v}^2} \bar{v}^2 \Delta\bar{v} \quad (29)$$

⁽²⁾ An approximate expression for the mean free path is $\bar{\ell} = \bar{v}_0/\bar{a}_0$ where \bar{v}_0 is kinematic viscosity. Free molecular flow theory applies when $\bar{\ell}/\bar{R}_0 = \bar{v}_0/\bar{a}_0 \bar{R}_0 \geq O(1)$.

Table I. Constants Defining Continuum and Free Molecular Expansion of Initially Uniform gas Clouds as $t \rightarrow \infty$ (See Eqs. 28 and 32)

	$\sigma = 0$				$\sigma = 1$				$\sigma = 2$			
	B	D	B'	D'	B	D	B'	D'	B	D	B'	D'
5/3	1	0.5	7.5	1.03	3	0.444	7.5	0.833	5	0.434	7.5	0.572
7/5	2	0.375	17.5	0.944	5	0.24	17.5	0.7	8	0.169	17.5	0.441
9/7	3	0.313	31.5	0.905	7	0.163	31.5	0.643	11	0.094	31.5	0.388
11/9	4	0.273	49.5	0.882	9	0.123	49.5	0.611	14	0.061	49.5	0.359
13/11	5	0.246	71.5	0.867	11	0.099	71.5	0.591	17	0.044	71.5	0.342

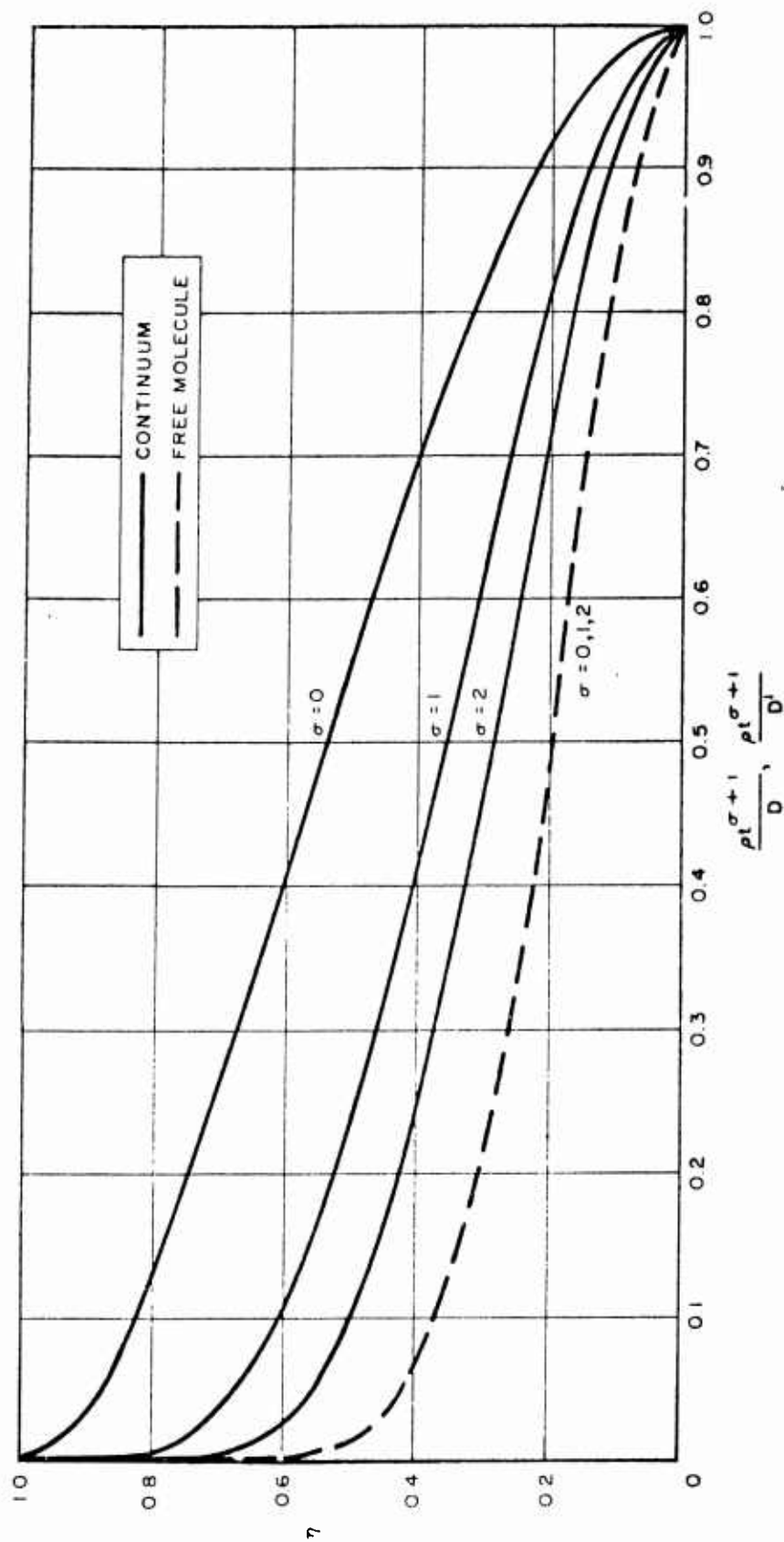


Figure 4. Variation of Density with η For Continuum and Free Molecular Expansion of Initially Uniform Gas ($t \rightarrow \infty$, $\gamma = 7/5$).

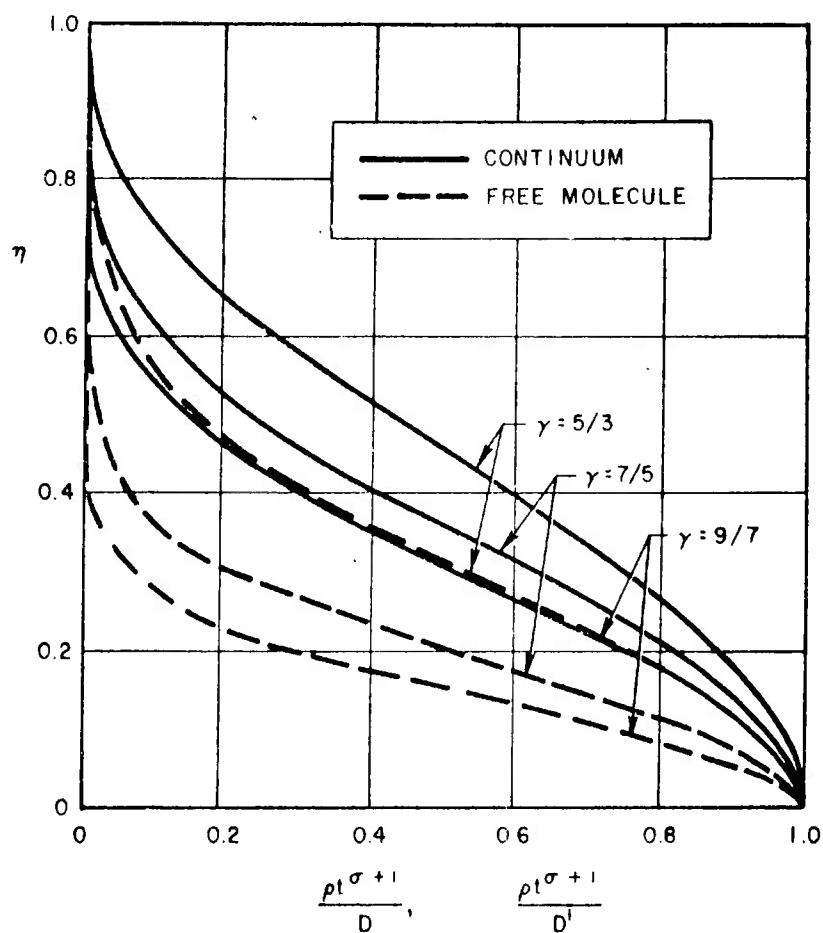


Figure 5. Variation of Density With η For Continuum and Free Molecular Expansion of Initially Uniform Gas ($t \rightarrow \infty, \sigma = 1$).

where $h = \gamma/(2\bar{a}_0^2)$. Assume that the container which confines \bar{m}_0 is removed at $\bar{t} = 0$. At a later time \bar{t} the particles in the velocity range \bar{v} to $\bar{v} + \Delta\bar{v}$ will be contained in the volume bounded by $\bar{r} = \bar{r}_0 + \bar{v}\bar{t}$ and $\bar{r} = \bar{r}_0 + (\bar{v} + \Delta\bar{v})\bar{t}$. This volume has the magnitude $\Delta\bar{V} = 4\pi\bar{v}^2\bar{t}^3\Delta\bar{v}$. The density of the particles $\bar{\rho}(\bar{r}, \bar{t}) \equiv \Delta\bar{m}_0/\Delta\bar{V}$ is then

$$\bar{\rho}(\bar{r}, \bar{t}) = \frac{(h/\pi)^{3/2} e^{-h(\frac{\bar{r}-\bar{r}_0}{\bar{t}})^2} \bar{m}_0}{\bar{t}^3} \quad (30)$$

Eq. 30 gives the density at (\bar{r}, \bar{t}) due to an initial elemental mass \bar{m}_0 at $(\bar{r}_0, 0)$. The expansion of a gas, with an arbitrary initial distribution of density, can be found by the linear superposition of the elementary solution given by Eq. 30 which corresponds to a spherical ($\sigma = 2$) expansion of an elemental three dimensional mass source. A similar expression can be found for an elemental line source ($\sigma = 1$) or plane source ($\sigma = 0$) located at $(\bar{r}_0, 0)$. The result, valid for all σ , is

$$\bar{\rho}(\bar{r}, \bar{t}) = \frac{(h/\pi)^{\frac{\sigma+1}{2}} e^{-h(\frac{\bar{r}-\bar{r}_0}{\bar{t}})^2} \bar{m}_0}{\bar{t}^{\sigma+1}} \quad (31)$$

where \bar{m}_0 equals the initial elemental mass for $\sigma = 2$, the elemental mass per unit length for $\sigma = 1$ and the elemental mass per unit surface area for $\sigma = 0$ (assuming a symmetrical expansion)

Now, consider the expansion of an initially uniform gas of radius \bar{R}_0 and $\sigma = 0, 1, 2$. After long times, the flow may be considered as having originated from an elemental source. The solution for $t \rightarrow \infty$ is then given

by Eq. 31 which can be put in non-dimensional form by introducing $r = \bar{r}/\bar{R}_0$, $v = \bar{v}/\bar{a}_0$, $t = \bar{t}\bar{a}_0/\bar{R}_0$, etc. In the present case define $\eta = (\gamma-1)r/2t$ so that $\eta = 1$ corresponds to the leading edge for the corresponding continuum expansion. (There is no leading edge for the free molecule expansion since $r \rightarrow \infty$ as $\rho \rightarrow 0$). Eq. 31 then becomes

$$\rho t^{\sigma+1} = D' e^{-B'\eta^2} \quad (32)$$

where

$$B' = 2\gamma/(\gamma-1)^2$$

$$D' = \sqrt{2\gamma/\pi}; \quad \gamma/2; \quad \frac{\gamma}{3}\sqrt{\frac{2\gamma}{\pi}} \quad \text{for } \sigma = 0, 1, 2$$

Note that B' is independent of σ . Numerical values for B' and D' are tabulated in Table I. Plots of $\rho t^{\sigma+1}$ versus η are presented in Figs. 4 and 5. Although the form of Eq. 32 is different from that of Eq. 25, the variation of $\rho t^{\sigma+1}$ with η is similar, for the two equations, except near the leading edge of the expansion. (See Figs 4 and 5.) For $\eta = 0$, Eq. 24 becomes $\rho t^{\sigma+1} = D$ and Eq. 32 becomes $\rho t^{\sigma+1} = D'$ so that D and D' determine the variation of ρ with t at $\eta = 0$. From Table I it is seen that for each σ and γ , $D < D'$ so that the density at the axis decays more rapidly, with time, for continuum flow. The difference in the expansion rates becomes more pronounced as γ approaches 1.

The later result can be more readily understood by considering the distribution of mass within the free molecular expansion. Let $\psi(\eta)$ be the

fraction of the total mass which is contained between the ordinate η and the axis. Then, using equation 32,

$$\psi(\eta) \equiv \frac{\int_0^r \rho r^\sigma dr}{\int_0^\infty \rho r^\sigma dr} = \text{erf}(\sqrt{B'}\eta) \quad \text{for } (\sigma = 0) \quad (33a)$$

$$= 1 - e^{-B'\eta^2} \quad (\sigma = 1) \quad (33b)$$

$$= \text{erf}(\sqrt{B'}\eta) - \frac{2\sqrt{B'}\eta}{\sqrt{\pi}} e^{-B'\eta^2} \quad (\sigma = 2) \quad (33c)$$

where $\text{erf}(\)$ is the error function. The effective leading edge of the free molecular expansion might be taken to correspond to $\psi(\eta) = 0.9, 0.99$ or 0.999 etc. The values of $\sqrt{B'}\eta$ which correspond to $\psi(\eta) = 0.9, 0.99$ and 0.999 are given in Table II. Let $\sqrt{B'}\eta \approx 2$ be taken as a characteristic value defining the effective leading edge of the free molecular expansion and let the corresponding value of r be denoted by R_{eff} . Then $R_{\text{eff}}/t \approx 2\sqrt{2/\gamma}$. This may be compared with $R/t = 2/(\gamma-1)$ for the continuum case, provided $\gamma \neq 1$ ⁽³⁾. Hence, the continuum expansion spreads more rapidly, particularly for γ near 1.

⁽³⁾ A more precise comparison can be made by computing $\psi(\eta)$ for the continuum flow. This can be done in closed form for $\sigma = 1$. In particular, for $\sigma = 1$ and continuum flow, Eqs. 25 and 28a show $\psi(\eta) = 1 - (1-\eta^2)^{(\gamma+1)/(\gamma-1)}$. For a given value of $\epsilon \equiv 1 - \psi(\eta)$ and $\sigma = 1$, the corresponding values of η for the free molecular and continuum flows are related by $\eta_{\text{F.M.}}/\eta_{\text{cont.}} = (\gamma-1) \sqrt{-\ln \epsilon} / \sqrt{2\gamma} \sqrt{1-\epsilon^{(\gamma-1)/(\gamma+1)}}$. For $\epsilon \ll 1$ and $\gamma \neq 1$, the latter gives $(R_{\text{eff}})_{\text{F.M.}}/(R_{\text{eff}})_{\text{cont.}} = (\gamma-1) \sqrt{-\ln \epsilon} / \sqrt{2\gamma}$ which is essentially the comparison given in the body of the report.

Table II. Effective Leading Edge of Free Molecular Expansion

$\psi(\eta)$	$\sqrt{B'} \eta$		
	$\sigma = 0$	$\sigma = 1$	$\sigma = 2$
0.9	1.16	1.52	1.77
0.99	1.82	2.15	2.40
0.999	2.33	2.63	2.85

HYPERSONIC JET

As noted in Ref. 5, the lateral expansion of a steady hypersonic jet, bounded by a vacuum, can be reduced to an equivalent unsteady expansion. This equivalence is now discussed.

Consider a nozzle to exhaust a hypersonic stream into a vacuum. Let \bar{u} be the velocity in the streamwise \bar{x} direction, \bar{u}_0 be the value of \bar{u} at the nozzle exit, $M_0 = \bar{u}_0 / \bar{a}_0$ and δ be a characteristic slope of the streamlines. If the dependent variables have the form

$$\bar{v} / \bar{u}_0 = O(\delta) \quad \bar{p} / \bar{p}_0 = O(1) \quad \bar{u} / \bar{u}_0 = 1 + O(\delta^2) \quad (34)$$

then the continuity and \bar{v} momentum equations reduce to Eqs. 1 (neglecting terms of order δ^2 compared with 1) with \bar{t} replaced by \bar{x} / \bar{u}_0 . These equations can be solved for \bar{v} and \bar{p} independent of the \bar{x} momentum equation. Bernoulli's equation can then be used to find \bar{u} (which equals \bar{u}_0 to order δ^2). The characteristic slope may be taken equal to that of the leading edge of the expansion after long times

$$\delta = \frac{2}{\gamma - 1} \frac{1}{M_0} \quad (35)$$

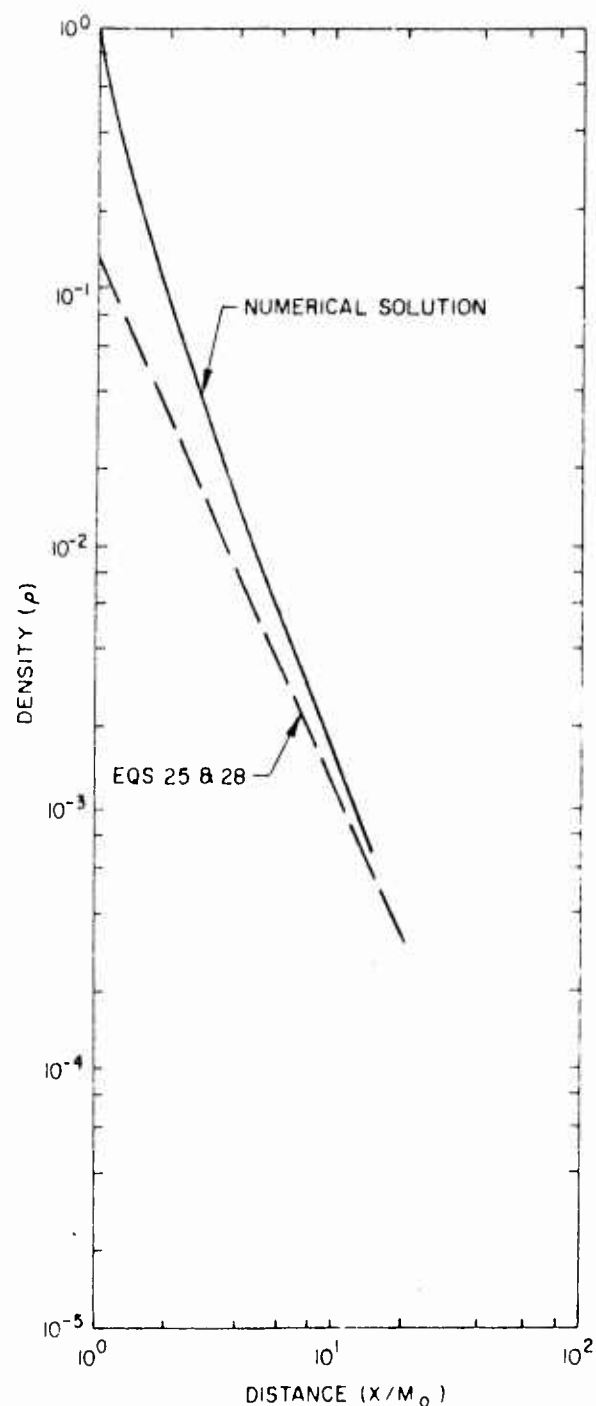
The above solution is self consistent when $M_0^2 \gg 1$ (provided γ is not nearly equal to 1) and hence applies to hypersonic jets. Thus, the previous unsteady solutions for $\sigma = 0, 1$ apply to two dimensional ($\sigma = 0$) and axisymmetric ($\sigma = 1$) hypersonic jets. The variable \bar{t} in the former is replaced by \bar{x}/\bar{u}_0 in the latter. If a non-dimensional $x = \bar{x}/R_0$ is introduced, then t is replaced by x/M_0 in the non-dimensional equations.

NUMERICAL SOLUTIONS

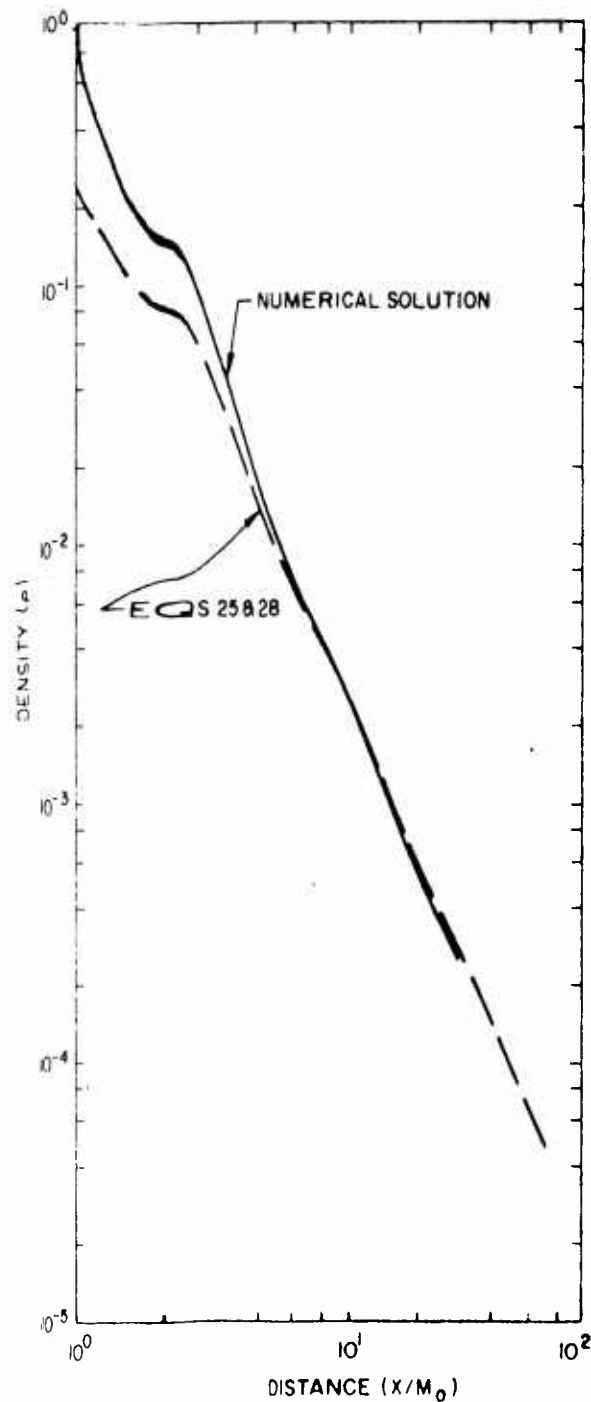
The equations of motion for an initially uniform hypersonic axisymmetric jet exhausting into a vacuum have been integrated numerically for $M_0 = 10, 15$ $\gamma = 1.4$ and $M_0 = 10, \gamma = 1.22$. The computations were performed by A. B. Troesch and R. F. Kramer using an existing digital computer program. The density along the axis of the jet was found as a function of distance and is plotted in Fig. 6. The approximate asymptotic solution, previously presented (Eqs. 28) is also noted in Fig. 6. It was not possible to continue the machine solution until the asymptotic form $\rho(x/M_0)^{\sigma+1} = D$ was attained.

Attempts to analytically extrapolate the machine results to very large distances, for direct comparison with the approximate theory, also met with some difficulty; the answer depending on the extrapolation scheme used. For the $M_0 = 10, 15; \gamma = 1.4$ case however, some confidence can be placed in the extrapolated value $D = 0.22$ which can be compared to the approximate value $D = 0.24$ given in Table I for $\gamma = 1.4, \sigma = 1$. As expected the results for $M_0 = 10, 15$ and $\gamma = 1.4$ were essentially independent of Mach number and are plotted as a single curve in Fig. 6b.

Reference 4 gives numerical results for the density at the center of an initially uniform spherical gas cloud expanding to a vacuum for $\gamma = 5/3$. The results are presented by a plot of ρ versus t which yields $D = 0.38$. The latter value can be compared with the approximate value $D = 0.432$ given in Table I for $\sigma = 2, \gamma = 5/3$. However, it should be noted that the initial portion of the continuum curve of ρ versus t in Ref. 4, is obviously incorrect since the numerical solution for ρ versus t approaches the asymptotic solution



a. Variation of Density with Distance along the Axis ($r = 0$) of Axisymmetric steady Hypersonic Jet ($M_0 = 10$; $\gamma = 1.22$) expanding into vacuum. Continuum Flow.



b. Variation of Density with Distance along the Axis ($r = 0$) of Axisymmetric steady Hypersonic Jet ($M_0 = 10, 15$; $\gamma = 1.4$) expanding into vacuum. Continuum Flow.

Figure 1

from below rather than from above (compare with Fig. 6). This casts some doubt on the correctness of the resulting value $D = 0.38$.

In general, it appears that the value of D given by the approximate solution (Eq. 28) tends to be somewhat too large for $\sigma = 1, 2$ although it is exact for $\sigma = 0$ and N an integer. The two numerical solutions noted herein indicated that D is too large by about 10%. Further numerical solutions are required to better define the accuracy of the approximate solution.

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